How to Learn the Basics of Derivatives in 10 Minutes or Less

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Abstract

This is a version of my in-class lecture on "doing derivatives" that I've been using since about 2003. I found over the years, that this material helps students at all levels learn enough basic calculus (particularly derivatives) for most uses in Economics at the undergraduate and MBA level.

1 Derivatives are most simply thought of as questions written in the "language" of math

You'll often see mathematical expressions like the following:

$$\frac{dX}{dY}$$

These expressions can always be read FROM THE BOTTOM UP. And the question is always this: "If I increase BOTTOM a little, how does TOP change?" Or, "If I increase BOTTOM a little, what happens to TOP?"

The little "d" stands for a tiny change (technically an "infinitesimally" tiny change). When we want to think of bigger changes, we tend to use a capital Greek "D", Δ.

For example, how would we write in math language "If I study a little more (say, an hour longer), what happens to my exam grade?". Well,

$$\frac{\Delta \text{Exam Grade}}{\Delta \text{Study}}$$

The "answer" could be simply that this expression is positive, i.e., greater than zero, meaning that if you increase your studying, your exam grade goes up (or, more dryly, that studying and grades on exams are positively related). For example,

$$\frac{\Delta \text{Exam Grade}}{\Delta \text{Study}} > 0$$

Of course, at some point, they are probably negatively related. Say you study 24 hours straight. At that point, if you study an extra hour, your exam grade might actually drop (if you don't get the chance to sleep before the exam). In this case we'd write,

$$\frac{\Delta \text{Exam Grade}}{\Delta \text{Study}} < 0$$

Even when the result of our derivative is an equation, we often are interested in plugging numbers into the equation to determine whether it is, overall, positive, negative or zero (i.e., increasing the BOTTOM will increase, decrease or not affect the TOP).

Finally, notice that while we often use symbols and such, you can really plug anything into this:

$$\frac{\Delta \text{My Weight}}{\Delta \text{Chocolate Chip Cookie Eating}} > 0$$

which says, if I eat more chocolate chip cookies, my weight increases.

We will use mostly symbols below. But I find that if students keep in mind the very basic idea that these are questions, read from the bottom up, it helps them a lot.

2   The Basic Rules of Derivatives

The basic rule of all derivatives is to "take the exponent down, put it in front and subtract one". Let's try this out on a few functions.

$$y = x^2$$

so, $\frac{dy}{dx}$ asks "if I increase x a little, what happens to y"? Applying our basic rule, we take the 2 down to the front and subtract one. I'll do that in steps here...

$$\frac{dy}{dx} = 2x^{2-1} = 2x^1 = 2x$$

What if we had $y = x^3$? Then we'd pull the 3 down and subtract 1.

$$\frac{dy}{dx} = 3x^{3-1} = 3x^2$$

What about this one $y = 2x^3$? Same thing but now when you pull the 3 down to the "front" you can either leave it as is or simplify the expression by multiplying it by the 2.

$$\frac{dy}{dx} = 3 \cdot 2x^{3-1} = 3 \cdot 2x^2 = 6x^2$$

What if it's not a number?! What if it's another variable like a "z" or a "w" or something?! ... Same thing...$y = zx^3$

$$\frac{dy}{dx} = 3 \cdot zx^{3-1} = 3zx^2$$

Actually, it's easier if it's a variable. Then I don't have to remember my multiplication tables. There are only two little notes here: (1) technically I should have used a curved "d" for partial derivative which is what we call a derivative when there are several variable options (here z or x) and we just picked one of them, x, and (2) all is okay as long as the other variable (here z) isn't "y" or some other function of x. But we can learn to handle those cases, they are just beyond the scope of this little helpful note.

What if the exponent isn't even a number? For example, $y = 2x^\alpha$.
Just apply the rule:

$$\frac{dy}{dx} = 2\alpha x^{\alpha-1}$$

Try this one on for size. Suppose $y = x^\alpha z^{1-\alpha}$. Let's take two different derivatives just to test our skills, with respect to x and with respect to z.

$$\frac{dy}{dx} = \alpha x^{\alpha-1} z^{1-\alpha}$$

and

$$\frac{dy}{dz} = (1-\alpha)x^\alpha z^{-\alpha}$$

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Again, it doesn't really matter. The rule is the same. Once we see what's on the "bottom" of the question (i.e., either "x" or "z" in this example), we know which exponent to pull down.

This last expression was no accident. It's actually the very common Cobb-Douglas production function which is widely used in economics. Here it is the way we normally write it.

$$Y = AK^{\alpha}L^{1-alpha}$$

where Y is production/output, K is the amount of capital used in production and L is the amount of labor used in production. Finally, the thing multiplying it all, A, is a way to capture technology.

The derivative of Y with respect to K (i.e., the question $\frac{dY}{dK}$) is what we call the Marginal Product of Capital, MPK and tells us "if I increase capital a little (i.e., marginally), by how much does production increase". Likewise, the derivative of Y with respect to L tells us the Marginal Product of Labor. Those are useful to know and calculate in economics. Try them both using your newly acquired derivative skills.

At his point I find that students can do all the basic derivatives they come across in basic economics classes. The final note however is on the one case that now screws them up. What is the derivative of y with respect to x now (i.e., $\frac{dy}{dx}$) ?

$$y = x$$

Well, don't panic. If we don't see the exponent there, it means it's a "1". And I'll remind you as well in advance that anything raised to the "zero" power is "1". So, use your rule:

$$\frac{dy}{dx} = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

which says (remember read from the bottom up) that a change in x causes a 1-for-1 change in y. Not to hard, right?